Lecture 8. (10/11/21)

- Finish Thm 1 + pf. from Lecture 7 notes.

Analytic functions.

Let $G \subseteq \mathbb{C}$ open, $f: G \to \mathbb{C}$.

Recall: • If we identify $\mathbb{C} \cong \mathbb{R}^2$ via $z = x+iy$, write $f = u+iv$, where u, v are real, then we have notions of partial derivatives $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, etc.

$$u_x(x_0, y_0) = \lim_{h \to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h}$$

$$u_y(x_0, y_0) = \lim_{k \to 0} \frac{u(x_0, y_0+k) - u(x_0, y_0)}{k}.$$

($\mathbb{R}$-diff)

- $f: G \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) if $\exists$ 2×2 matrix A s.t.

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0+h, f(y_0+k) - f(x_0,y_0) - A\binom{h}{k}|}{\sqrt{h^2+k^2}} = 0.$$

If A exists, $A = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$, but $\mathbb{R}$-diff. is stronger than existence of partial derivatives.

Back to $\mathbb{C}$, where we also have multiplicative structure: given $z, w \in \mathbb{C}$, $zw \in \mathbb{C}$.

<u>Def.</u> ① For $f: G \to \mathbb{C}$, f is <u>$\mathbb{C}$-differentiable</u> at $z_0 \in G$ if

$$f'(z_0) = \lim_{\zeta \to 0} \frac{f(z_0+\zeta) - f(z_0)}{\zeta} \quad \text{exists.}$$

<u>complex derivative</u> of f at z_0.

<u>Prop 0</u>. If f is $\mathbb{C}$-diff. at z_0, then f is diff. at (x_0, y_0) as $f: \mathbb{R}^2 \to \mathbb{R}^2$.

<u>Pf</u>. The multiplication map $\zeta \to f'(z_0)\zeta$ is linear on $\mathbb{C} \cong \mathbb{R}^2$, since these are equal as vector spaces. Thus, $\exists$ 2×2-matrix A s.t. $f'(z_0)\zeta = A\binom{h}{k}$ if $\zeta = h + ik$; in fact, if $f'(z_0) = \alpha + i\beta$, then

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Now, existence of $f'(z_0)$ is equivalent to

$$\lim_{\zeta \to 0} \frac{|f(z_0+\zeta) - f(z_0) - f'(z_0)\zeta|}{|\zeta|} = 0,$$

which proves differentiability since $\mathbb{C} \cong \mathbb{R}^2$ also as metric space! ☒

<u>Rem.</u> We will come back to the calculation $A \longleftrightarrow f'(z_0)$ to find Cauchy-Riemann equations.

<u>Def.</u> ② $f: G \to \mathbb{C}$ is <u>analytic</u> or <u>holomophic</u> in G if f is <u>cont. diff.</u> (u_x, u_y, v_x, v_y exist and are cont. in G) and $\mathbb{C}$-diff. at every $z_0 \in G$. (cont. diff $\underset{\text{notation}}{\Rightarrow}$ C^1)

- Same rules apply to $\mathbb{C}$-der. as to $\mathbb{R}$-der: linearity, product rule, chain rule, ...

<u>Prop 1.</u> Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have $R > 0$ and $G = B(a, R)$.
Then f is analytic in G. Moreover,

(i) $\sum_{n=0}^{\infty} n(n-1)\dots(n-k+1)\, a_n (z-a)^{n-k}$

has ROC $= R$ and $= f^{(k)}(z_0)$, for any $k \geq 1$.

(ii) $a_n = \frac{1}{n!} f^{(n)}(z_0)$.

<u>Pf.</u> If we show (i) for $k=1$, it follows for all k by induction. This $\Rightarrow$ f has ∞ many $\mathbb{C}$-der. at every $z \in G$ ($\Rightarrow$ in part., f is C^1) $\Rightarrow$ f is analytic.

<u>Let's prove (i)</u> for $k=1$. Set $b_{n-1} = n a_n$ for $n = 1, 2, \dots$, and consider p.s.

$$\sum_{k=0}^{\infty} b_k (z-a)^k = \sum_{n=0}^{\infty} n a_n (z-a)^{n-1}.$$

Let $R' =$ ROC for and compute

$$\frac{1}{R'} = \limsup_{k\to\infty} |b_k|^{1/k} =$$

$$\limsup_{k\to\infty} (k+1)^{1/k} |a_{k+1}|^{1/k}.$$

Calculus $\Rightarrow$ $\lim_{k\to\infty} (1+k)^{1/k} = 1$ $\overset{\text{(Ex.)}}{\Longrightarrow}$

$$\limsup_{k\to\infty} (1+k)^{1/k} |a_{k+1}|^{1/k} = \limsup_{k\to\infty} |a_{k+1}|^{1/k}$$

$$= \limsup_{n\to\infty} |a_n|^{\frac{1}{n-1}} = \limsup_{n\to\infty} \left(|a_n|^{\frac{1}{n}}\right)^{\frac{n}{n-1}}.$$

Denote $s_n = |a_n|^{1/n}$. By def. $\exists$ subseq. such that $s_{n_k} \to \frac{1}{R}$, which implies

$$(s_{n_k})^{\frac{n_k}{n_k-1}} \to \frac{1}{R}.$$

But $\sup_{l \geq n} s_l^{l/l-1} \geq s_{n_k}^{n_k/n_k-1}$ when $n \leq n_k \Rightarrow$

$$\limsup_{n\to\infty} s_n^{n/n-1} \geq \lim_{k\to\infty} s_{n_k}^{n_k/n_k-1} = \frac{1}{R}$$

$$\Rightarrow \frac{1}{R'} \geq \frac{1}{R}.$$

For opposite inequality, note for any $\varepsilon > 0$ $\exists N$ s.t. $s_n \leq \frac{1}{R} + \varepsilon$, $n \geq N$.

$$\Rightarrow s_n^{n/n-1} \leq \left(\frac{1}{R} + \varepsilon\right)^{n/n-1} \to \frac{1}{R} + \varepsilon$$

Thus, $\limsup_{n\to\infty} s_n^{n/n-1} \leq \limsup_{n\to\infty} \left(\frac{1}{R} + \varepsilon\right)^{n/n-1}$

$$= \lim_{n\to\infty} \left(\frac{1}{R} + \varepsilon\right)^{n/n-1} = \frac{1}{R} + \varepsilon.$$

Thus, $\frac{1}{R'} \le \frac{1}{R}+\varepsilon$, where $\varepsilon>0$ is arbitrary.

Consequently $\frac{1}{R'}=\frac{1}{R}$, which in turn proves that ROC for $\sum_{n=0}^{\infty} n a_n (z-a)^n$ is R.

We must now show that for $|\overset{\text{fixed}}{z_0}-a|<R$

$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ has $\mathbb{C}$-der. at z_0 s.t.

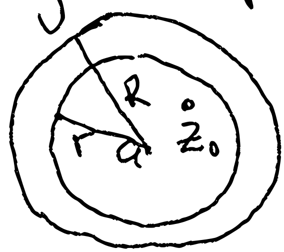
$$f'(z_0)=\sum_{n=0}^{\infty} n a_n (z_0-a)^n.$$

<u>Claim</u> (DIY). $\frac{d}{dz}(z-a)^n = n(z-a)^{n-1}$.

Set $P_n=\sum_{k=0}^{n} a_k(z-a)^k$. Then, $P_n'=\sum_{k=0}^{n} k a_k(z-a)^k$

and for $|z-a|\le r<R$, $P_n\to f$ unif.

and $P_n'\to g$ unif., where



$|z_0-a|<r$

We claim $\lim_{\zeta \to 0} \dfrac{f(z_0+\zeta)-f(z_0)}{\zeta} = g(z_0)$.

What happens in diff. quot. when we replace f by P_n? For simplicity, assume $a=0$. Take $|\zeta|$ small so $|z_0+\zeta|<r$

$$f(z_0+\zeta)-f(z_0)-\left(P_n(z_0+\zeta)-P_n(z_0)\right)$$

$$= \sum_{k=n+1}^{\infty} a_k\left((z_0+\zeta)^k - z_0^k\right).$$

Moreover, $x^k - y^k = (x-y)(x^{k-1}+x^{k-2}y+\dots+y^{k-1})$

$\Rightarrow \left|(z_0+\zeta)^k - z_0^k\right| \le |\zeta| \cdot k r^{k-1}$. Thus,

$$\frac{\left|f(z_0+\zeta)-f(z_0)-\left(P_n(z_0+\zeta)-P_n(z_0)\right)\right|}{|\zeta|} \le$$

$$\sum_{k=n+1}^{\infty} |a_k| \cdot k r^{k-1} \to 0$$ unif. in ζ, since

ROC of $\sum_0^\infty k a_k z^{k-1}$ is $R > r$.

Thus, fix $\varepsilon > 0$,

$$\left| \frac{f(z_0+\zeta)-f(z_0)}{\zeta} - g(z_0) \right| \le$$

$$(1) \left| \frac{f(z_0+\zeta)+f(z_0)}{\zeta} - \frac{P_n(z_0+\zeta)+P_n(z_0)}{\zeta} \right| +$$

$$(2) \left| \frac{P_n(z_0+\zeta)-P_n(z_0)}{\zeta} - P_n'(z_0) \right| +$$

$$(3) \left| P_n'(z_0) - g(z_0) \right| .$$

By above and unif. conv. $P_n \to g$, $\exists n_0$ s.t. term (1) $< \varepsilon/3$, (3) $< \varepsilon/3$. Finally, since $\mathbb{C}$-der of P_{n_0} exist and $= P_{n_0}'$ $\exists \delta > 0$ s.t. if $|\zeta| < \delta$ then (2) $< \varepsilon/3$.

$$\Rightarrow \left| \frac{f(z_0+\zeta)-f(z_0)}{\zeta} - g(z_0) \right| < \varepsilon \Rightarrow \text{(i) is proved.}$$

(ii) follows from (i) by setting $z=a$. $\blacksquare$

$$= \limsup_{n\to\infty} |a_n|^{1/n-1} = \limsup_{n\to\infty} \left(|a_n|^{1/n}\right)^{\frac{n}{n-1}}.$$

Let $s_n = \sup_{k \geq n} |a_k|^{1/k}$. Then s_n is a decreasing seq. of ≥ 0 numbers s.t. $s_n \searrow 1/R$ by def.. We want to compute $\lim_{n\to\infty} s_n^{1+\frac{1}{n-1}}$. We leave $R = 0, \infty$ as DIY. Thus, assume $\lim_{n\to\infty} s_n = \frac{1}{R}$ $(\neq 0, \infty)$. Then $\exists N$ s.t. $\frac{1}{R} \leq s_n < \frac{2}{R}$ for $n \geq N$. But $s_n^{1+\frac{1}{n-1}} = s_n \cdot (s_n)^{1/n-1}$ and

$$\left(\frac{1}{R}\right)^{\frac{1}{n-1}} \leq (s_n)^{1/n-1} \leq \left(\frac{2}{R}\right)^{\frac{1}{n-1}}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$1 \qquad\qquad 1 \qquad\qquad 1$$